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Singular cross sections in muon colliders

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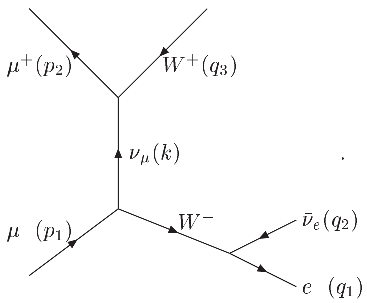
Abstract. We address the problem that the cross section for the collisions of unstable particles diverges, if calculated by standard methods. This problem is considered for beams much smaller than the decay length of the unstable particle, much larger than the decay length and finally also for pancake- shaped beams. We find that in all cases this problem can be solved by taking into account the production/propagation of the unstable particle and/or the width of the incoming wave packets in momentum space.

1 Introduction

When one applies the Feynman Rules and the Golden Rule to a collision of unstable particles, the cross section turns out to diverge. We summarize how this happens. The divergence occurs for instance in the graph



. (1)

The lower half of this diagram looks like the decay of the muon. The consequence is that the momentum k may be on its mass shell. After all, that is what one gets from the decay of a muon: a muon neutrino on its mass shell. The propagator of this muon neutrino contains a factor

$$\Delta(k) = \frac{i}{k^2 + i\epsilon} \qquad (2)$$

that gets squared and integrated over as the Golden Rule tells us to do. Because the neutrino that contributes the just given factor can be on its mass shell, we find that in lowest order in ϵ the result will go as ϵ^{-1}. This is a divergent quantity since ϵ is taken to be infinitesimal. This problem is of relevance for muon colliders as was already noted in [1].

The question that naturally poses itself is if the cross section really diverges, and if so, how this can be regularized, and if not, how we can compute it. In general we can say that there are two possibilities to solve the question of the divergent cross section. The first is that one takes into account that unstable particles cannot really be in or out states. This can be done by considering Feynman graphs that include the production process of the unstable particle. The second possibility is that one takes into account that the incoming wave packets are not really sharp momentum states but that we always have interference between states with the same total momentum but with some of the momentum moved from one incoming particle to the other. This is of importance if the peak structure of the matrix element is sharper than the size of the incoming wave packet in momentum space. We indeed have this in the case of the unstable particle, since the propagator that causes the divergence has no particular size attribute, except for the ϵ, which is taken to be infinitesimal.

This problem has mainly been discussed for the case that would be applicable to, for instance, muon colliders. Cf., [2, 3, 6]. We will mostly address a case of academic interest that has been considered alongside the realistic case, namely that of very wide beams (i.e., much larger than the decay length of the muon). We will try to apply the methods applicable to the realistic case also in the case of infinitely wide beams. For this we will need some modifications to the realistic case. This case is considered in the next section. We reconsider it because we use different, more covariant notations, best introduced in a somewhat more familiar setting, and also because we have some points to add.

Finally we consider the case of pancake-shaped beams (i.e., much larger than the decay length in the transversal directions and much smaller than that in the longitudinal direction).

2 The realistic case

Here we discuss the solution as would be applicable in muon colliders, that is, for beams of which the size is much

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smaller than the decay length of the unstable particle. In this case the propagator of the unstable particle confines its momentum to the mass-shell and because production and collision can be chosen to be well-separated in space, we need not worry about the production process. There is, however, still the point that the matrix element is peaked sharply enough to notice interference between states where some momentum is shifted from one incoming particle to the other. This case has been solved in [2], however we again present this here using notations that are manifestly covariant.

The quantum distribution function $n(p,r)$, introduced in [3], is used to describe the particles. It is defined to be given by

$$n(p,r) = \frac{m}{(2\pi)^3} \int d^4\varDelta\, \delta(p\cdot\varDelta) \times \phi(p+\tfrac{1}{2}\varDelta)\, \phi(p-\tfrac{1}{2}\varDelta)^*\, e^{-i\varDelta\cdot r}. \quad (3)$$

The $\delta(p\cdot\varDelta)$ is used to confine the particle to its mass shell. This implies that we use the approximation that the components of $\varDelta$ are much smaller than the ones of p. The quantum distribution function contains as much information about the state of a particle as a density matrix. From it the probability densities in momentum and position space can be found. They are given by

$$\rho(p) = \frac{p^0}{m}\int d^3r\, n(p,r); \qquad \rho(r) = \int \tilde{d}p \frac{p^0}{m} n(p,r). \quad (4)$$

Both are (in the approximation that they are sharply peaked in momentum space) zeroth components of four-vectors. This should, of course, be the case with densities. The probability measure that belongs to them is respectively $\tilde{d}p$ (which is by definition equal to $\frac{d^3p}{(2\pi)^3 2p^0}$) and d^3r. This looks pretty much as if $n(p,r)$ were a joint probability for r and p, but of course such a thing cannot really exist in quantum mechanics and actually $n(p,r)$ does not need to be a real function, which discards it as a probability density.

We extend the definition of the luminosity to

$$dL(p_1,p_2,\rho) = \tilde{d}p_1\, \tilde{d}p_2 \frac{\sqrt{-\mathrm{GD}}}{m_1 m_2} \times \int d^4r\, n_1(p_1,r) n_2(p_2, r+\rho). \quad (5)$$

This reduces to the normal definition for dL by setting $\rho = 0$. The GD that occurs here stands for "Gramm determinant" and is equal to $p_1^2 p_2^2 - (p_1\cdot p_2)^2$. Also the definition of the cross section is extended, namely to

$$d\sigma(p_1,p_2,\varDelta) = \frac{1}{4\sqrt{-\mathrm{GD}}}(2\pi)^4 \delta(p_1+p_2-q)\, \tilde{d}q_1 \cdots \tilde{d}q_n \times \mathcal{M}(p_1+\tfrac{1}{2}\varDelta, p_2-\tfrac{1}{2}\varDelta, q) \times \mathcal{M}(p_1-\tfrac{1}{2}\varDelta, p_2+\tfrac{1}{2}\varDelta, q)^*. \quad (6)$$

This gives the well-known definition for the cross section if we take $\varDelta = 0$. The exact expression for the number of events W (see for instance [5]) can, using these definitions, be written as

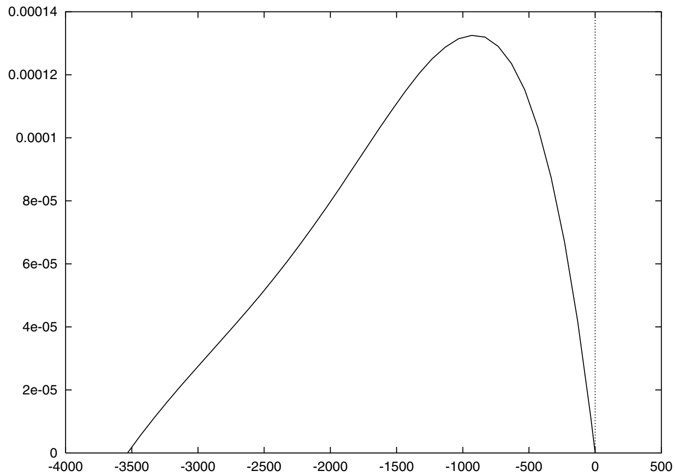

Fig. 1. $d\sigma/dt$ (fb/GeV^2) vs. t (GeV^2) for $\sqrt{s} = 100\,\mathrm{GeV}$

$$W = \int dL(p_1,p_2,\rho) \frac{d^4\varDelta}{(2\pi)^4} d^4\rho\, e^{-i\varDelta\cdot\rho} d\sigma(p_1,p_2,\varDelta). \quad (7)$$

Normally one assumes that $d\sigma$ does not vary much with $\varDelta$ and for that reason it is safe to put $\varDelta = 0$. Then the integrals over $\varDelta$ and ρ become trivial and the familiar result that "number of events is cross section times luminosity" is obtained. In our case, however, σ has a pole at $\varDelta = 0$ so this approximation cannot be made.

The reasoning about how to get a finite cross section is essentially identical to what is given in [2], so we will not present it here. In the end the result is

$$\sigma = a\pi \int d\sigma_{\mathrm{red}} \frac{1}{|k_\perp|} \delta(k^2 - m^2), \quad (8)$$

where the "red" in $d\sigma_{\mathrm{red}}$ stands for "reduced" meaning that the factors $1/(k^2 \pm i\epsilon)$ that cause the divergence have to be left out. a is the transverse size of the beam. We take this to be given by $a = \sqrt{\pi}\,\sigma = \sqrt{\pi}\cdot 10\,\mu\mathrm{m}$, where (for round beams) σ is the standard deviation in the position of the particles in the beam in either direction perpendicular to the beam. This was also done in [2]. Because this part of the cross section is proportional to this size, the effect is called "linear beam size effect". If there are contributions to the cross section from parts of phase space away from the singularity they should be added separately to the cross sections. These parts do not depend on the beam size. Below we will see that such a separation arises rather naturally from the shape of the graph of $d\sigma/dk^2$. The part of the cross section that has to do with regions of phase space away from the singularity will be called the "regular cross section", while the part that comes from regions near the singularity (or singularities) will be called the "semi-singular cross section".

As in [2], we consider the process $\mu^- + \mu^+ \to W^+ + e^- + \bar{\nu}_e$. For $d\sigma/dt$ at 100 GeV we find the graph in Fig. 1. The infinite spike at $t = 0$ is caused by the instability. Figure 2 shows a detail of the same graph. Now the singularity is

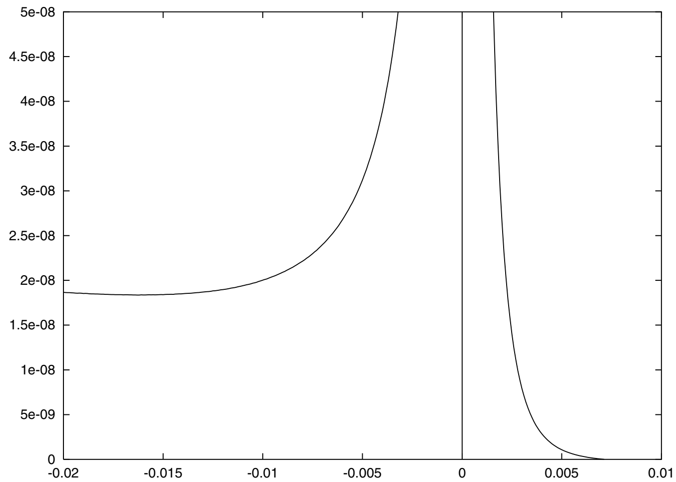

Fig. 2. $d\sigma/dt$ (fb/GeV^2) vs. t (GeV^2) for $\sqrt{s} = 100\,\text{GeV}$ around $t = 0$. This is a detail of the previous plot

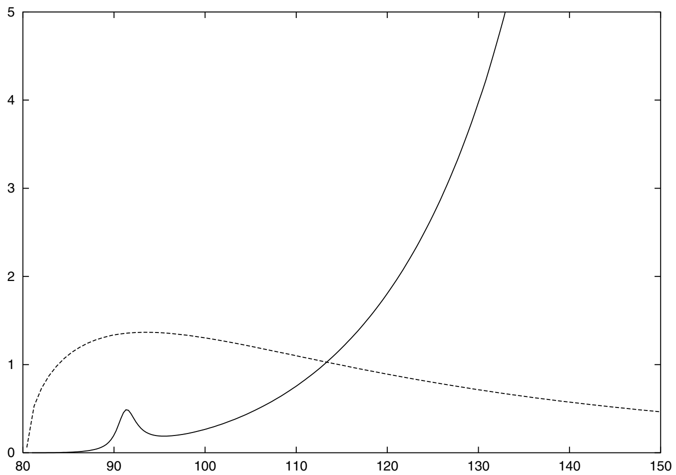

Fig. 3. σ (fb) vs. $\sqrt{s}$ (GeV). The solid line is the regular cross section and the dashed one is the pseudo-singular cross section. $a = \sqrt{\pi} \cdot 10\,\mu\text{m}$

prominently present. The reader can compare the numbers on the axes of both graphs to get an idea how narrow the singularity actually is. To be able to calculate the regular part of the cross section without having to worry about the singularity, the cut $t < -m_\mu^2$ is used (as was done in [2]). The two graphs just shown, justify this cut (i.e., $m_\mu^2 \sim 0.01\,\text{GeV}^2$ which is in the neighbourhood of the minimum of $d\sigma/dt$).

The total cross section for pseudo-singular and regular cross sections are plotted in Fig. 3. For the regular cross section we find the same graph as in [2], but for the semi-singular cross section our graph is about a factor 1.7 higher. The reason for this difference appears to be twofold. In the first place, we did our calculations from standard model coupling constants, while [2] expresses the cross section in other decay widths and cross sections. If we take this into account, the factor 1.7 becomes a factor 2. This factor 2 is then due to an error in equation 46 in [2]. This equation should have an extra factor 2 on the right hand side.

Consequently we find that the semi-singular cross section dominates up to about 113 GeV. [2] has 105 GeV. Furthermore it should be noted that at about 90 GeV the semi-singular and regular cross sections are about the same order of magnitude because the regular one has a peak there, caused by the Z particle. For $\sqrt{s}$ a bit above threshold the semi-singular cross section dominates strongly. At about, say, $\sqrt{s} \sim 85\,\text{GeV}$ one can safely forget about the regular part. Above $\sqrt{s} \sim 150\,\text{GeV}$ the pseudo-singular cross section does not play any role anymore. This is the case up to arbitrary high energies because asymptotically the semi-singular cross section goes down as $1/(s\sqrt{s})$, while the regular one goes down as $1/s$.

To calculate these cross sections, six diagrams involving γ, $W^\pm$ and Z^0 as fundamental bosons were added. The algebra necessary was done by the C++ computer algebra library GiNaC which is described in [9]. After that the integrations were carried out by adaptive Simpson integration. Unstable intermediate particles were given propagators using the $iM\Gamma$ prescription. This, of course, raises the issue of gauge invariance. It was checked that for high energy the cross section goes down as $1/s$, but there may still be unnoticed terms suppressed by a factor Γ^2/M^2 that did not show up in these calculations. The authors hope to address these issues in a subsequent paper.

In practice one does not need the cut-off $t < -m_\mu^2$. This is because this cut-off is already implied by the cut-offs imposed by measurability. If we take $\sqrt{s} = 100\,\text{GeV}$ and demand that the energy of the outgoing electron of the pseudo-singular process is at least 1 GeV and that the angle under which this electron appears is at least 2° we have made it impossible for the muon neutrino to be on its mass shell (actually we then find that t indeed is always negative and has an upper limit of about $-5m_\mu^2$).

However, if one wants to include the linear beam size effect in the normal Monte Carlo integration procedure, one can do this by doing the replacement

$$\frac{i}{k^2 + i\epsilon} \to \frac{i}{k^2 + i|k_\perp|/a}. \tag{9}$$

This, in spite of its ad-hoc appearance, gives precisely the correct answer. It effectively introduces a decay to the muon neutrino to account for the fact that it can no longer collide after it has left the beam. This way of handling the divergence supposes that number of particles in the beam goes to zero exponentially as we leave the beam (i.e., the $iM\Gamma$ prescription is about exponential decay), which is not terribly realistic, but in most cases this will not matter.

3 Infinitely wide beams

3.1 The wrong way

The method of calculation introduced in [4] (to be called Ginzburgs method in the rest of this section) starts by observing that the propagator of an unstable particle is given by $(p^2-m^2+im\Gamma)^{-1}$. We then "conclude" that the mass of the unstable particle has acquired an imaginary part. The four momentum-squared of this particle should be complex too. In its rest frame it is given by

$$p_1 = (m - i\Gamma/2, \mathbf{0}). \tag{10}$$

We now write for the "new" value of k^2, that is the value that takes the complex momentum components into account,

$$k^2_{\text{new}} = (p_1 - q)^2 = m^2 - im\Gamma + q^2 - 2(m - i\Gamma)q^0, \tag{11}$$

where $q = q_1 + q_2$ as drawn in (1). For some reason we take the same values for the components of q but only change the components of p_1 and k. Normally (i.e., without complex momentum components), the value of q^0 is given by

$$q^0 = \frac{m^2 + q^2 - k^2}{2m}. \tag{12}$$

After substituting this, and neglecting the small quantity proportional to $k^2\Gamma$, we obtain

$$k^2_{\text{new}} = k^2_{\text{old}} - i\frac{\Gamma}{2m}(m^2 - q^2). \tag{13}$$

This complex value then replaces the one given in (2), and a finite result is obtained.

The problem with all this is, of course, that it is not terribly difficult to think of a process where one of the outgoing/incoming particles involved is unstable and then an incoming complex momentum component has, by momentum conservation, no place to go. Significantly, we never hear about momentum conservation for the other vertex in the diagram.

In [6] it is shown that the result obtained by using the here described method is exactly what one would expect for a muon that decays in a medium of anti-muons. This result is

$$\sigma = \int (1 - \cos\theta) w(\omega)\, d\omega \frac{\sin\theta\, d\theta\, d\phi}{4\pi} \sigma_{\nu\mu\to W}(s_{\nu\mu}). \tag{14}$$

The factor $w(\omega)\, d\omega\, \sin\theta\, d\theta\, d\phi/4\pi$ is the probability measure of finding a muon neutrino with a certain momentum. This calculation is done in the rest frame of the decaying muon. However, three remarks are in order here[1].

1. To obtain this result, the definition of the quantity cross section has to be modified;
2. The same modification of definition can be aplied to our result (i.e., (26), to be derived in the next section) and the result will be the same;
3. It is a bit of a coincidence that the modified cross section of [6] turns out to have the same value as Ginzburgs method

[1] An email discussion with V.G. Serbo was helpfull to get these points completely clear

Let us discuss these points in this order.

Firstly **(1)**, the number of events W is related to the cross section via

$$W = V_4 \sqrt{(J_1 \cdot J_2)^2 - J_1^2 J_2^2}\, \sigma_{\mu\mu}, \tag{15}$$

where V_4 is the four-volume in which the beams overlap and $J_{1,2}$ denotes the four-flux of the two beams. The space integral over J_1 is the number-of-particles four-vector N_1^μ, defined by

$$N^\mu = N u^\mu, \tag{16}$$

with N the number of particles. We thus have

$$\frac{dW}{dt} = \sqrt{(N_1 \cdot J_2)^2 - N_1^2 J_2^2}\, \sigma_{\mu\mu}. \tag{17}$$

On the other hand, we expect to be able to calculate the number of events from considering collisions between muon neutrino's and muons, taking the momentum distribution of the muon neutrino's into account. Doing so we find

$$\frac{dW}{dt} = \int w(\omega)\, d\omega \frac{\sin\theta\, d\theta\, d\phi}{4\pi} \times \sqrt{(N_\nu(\omega) \cdot J_2)^2 - N_\nu(\omega)^2 J_2^2}\, \sigma_{\nu\mu}(s(\omega))\ . \tag{18}$$

After specializing to the rest frame of the μ^- we see that the division of the two flux factors gives the $1-\cos\theta$ (For this one has to assume that $k^2 = p_2^2 = 0$.) so that we indeed find (14) back. The modification to the definition of the cross section is that one equates these two dW/dt's. The first dt refers to the time the μ^- track spends in the μ^+ cloud while the second dt refers to the time the decay product spends in this cloud. In a μ^+ cloud of infinite size equating these two indeed would seem to be the only thing that could give a finite result. However, one should realize that for any cloud of particles of finite size the standard definition of the cross section involves and integral over time and then the quotient of the two just mentioned times will appear in the result. This will depend on beam shapes.

Secondly **(2)**, also our result (i.e., (26)), to be derived in the next section, contains a factor d^4r. One could also pull a factor dt out of this and obtain exactly the same result as by Ginzburgs method. So, whether or not one likes the equating of dW/dt's mentioned in the foregoing point, one does not need complex momentum conservation to obtain the result that belongs to it.

Thirdly **(3)**, the matrix element, as seen in position space, used in Ginzburgs method, is not as advertised in [6]. To see this, we should realize that the method of Ginzburg only modifies the propagator of the muon neutrino. In particular, nothing is altered in the prescription of removing external propagators. The consequence of this is that the number of muons does not decrease. After all, this prescription was designed to describe a stable particle that comes in from infinity. This means that the production rate of neutrino's inside the μ^+ cloud is given (in the rest frame of the μ^-) by

$$\frac{dN_{\nu_\mu}}{dt} = N_\mu \Gamma_\mu\ . \tag{19}$$

Thus, every muon produces a large number of neutrino's. Because of the conservation of complex momenta, the muon neutrino gets a decay time equal to the decay time one would expect for the muon. The consequence is that the number of muon neutrino's at a particular time is given by

$$N_{\nu_\mu}(t) = \int dt'\, \theta(t-t')\, e^{-\Gamma_\mu (t-t')} \frac{dN_{\nu_\mu}}{dt'} = N_\mu. \qquad (20)$$

So in the method of Ginzburg the number of neutrino's is equal to the number of muons not because every muon decays into a single muon neutrino but because the decay constant of the muon neutrino is artificially made equal to its production constant. That this gives the same result as equating the two dW/dt's as discussed under point **(1)** can be easily understood because the muon neutrino's now only exist near the path of the μ^-, so the two dt's now both refer to the time the μ^- spends in the μ^+ cloud.

3.2 The right way

Here we take the production process of the unstable particle into account. This is reasonable because unstable particles cannot really be asymptotic states. In the context of scalar fields, Veltman (cf., [7]) has shown that if one takes only stable particles for asymptotic states, one gets a unitary S-matrix. Thus we must describe the unstable particle as an internal line of a larger Feynman graph that includes the production process. Consequently its momentum can have four independent components. Of course, we really do not want to include much information about the production process of the unstable particle in our calculations. For this reason we define the wave function of the unstable particle to be given by

$$\psi(p) = \sqrt{\frac{2}{m_1 \Gamma}} \int \tilde{d}p_a\, \tilde{d}p_b\, \phi_a(p_a)\phi_b(p_b) \times (2\pi)^4 \delta^4(p_a + p_b - p)\, \mathcal{M}(p_a, p_b, p), \qquad (21)$$

where $\mathcal{M}$ is the matrix element (or perhaps the sum of matrix elements) that describe the production process. This definition assumes that the unstable particle in produced in a two-to-one collision, however we can safely include terms where the production process has as many in- or outgoing particles as we would like. The outgoing particles are represented by complex conjugates of the wave functions that they are measured to be in. Such a measurement should be carried out in order to make sure that the momentum of the unstable particle is fixed very accurately. When doing a calculation with this we still have to include the propagator of the unstable particle. The just given wave function only replaces the part of the matrix element that describes the production process, not the propagation of the unstable particle. For this propagator, the replacement

$$\frac{i}{p^2 - m^2 + im\Gamma} \to \frac{1}{m\Gamma} \qquad (22)$$

should be used, since the momentum of the unstable particle is assumed to be fixed—by the just-described measurement procedure—on its mass shell so accurately that a function of which the width is of the order Γ (as the propagator is) cannot notice the difference.

The normalization of this wave function, as occurs in its definition above, was obtained by considering the production process followed by the decay process. The number of events in this is demanded to be unity because we choose our wave function to describe one unstable particle.

In this case the definition of the quantum distribution function should be modified a bit. It becomes

$$n(p,r) = \int \frac{d^4\Delta}{(2\pi)^4} e^{-i\Delta\cdot r} \psi(p + \tfrac{1}{2}\Delta)\psi(p - \tfrac{1}{2}\Delta)^*. \qquad (23)$$

Probability densities in momentum and position space can be obtained from this. They are given by

$$\rho(p) = \int d^4r\, n(p,r); \qquad \rho(r) = \int \frac{d^4p}{(2\pi)^4}\, n(p,r). \qquad (24)$$

The luminosity is now defined by

$$dL(p_1,p_2,\rho) = \frac{\sqrt{-\mathrm{GD}(p_1,p_2)}}{m_1 m_2 \Gamma} \frac{d^4p_1}{(2\pi)^4} \tilde{d}p_2 \times \int d^4r\, n_1(p_1,r) n_2(p_2, r+\rho). \qquad (25)$$

This is for the case of a stable particle colliding with an unstable one. The reader will presumably have no difficulty figuring out what to use for two unstable particles if he compares this to (5). The cross section is defined in exactly the same way as in Sect. 2. Furthermore, the just given definition for the luminosity was chosen in such a way that (7) is kept the same too.

After having introduced this, the calculation of the number of events proceeds along exactly the same lines as in [2] and we will not present it here. The result is

$$W = \frac{1}{m_2} \int \frac{d^4p_1}{(2\pi)^4} \tilde{d}p_2\, \tilde{d}k\, d^4r \times \int_0^\infty d\alpha\, n_1(p_1,r) n_2(p_2, r+\alpha k) f_s(k|p_1) \times \sqrt{(k\cdot p_2)^2 - k^2 p_2^2}\, \sigma(k,p_2). \qquad (26)$$

The interpretation of the above formula is that the unstable particle decays at position r, the decay product has momentum k and then collides with the other particle at position $r + \alpha k$, where α is a positive number, as one would expect. If we specialize to the case where the longitudinal beam size is much larger than the transversal one, we find back formula (8). The densities that occur here should be considered to be densities of decay events. It is perfectly natural that decay events are characterized by four co-ordinates.

4 Pancake-shaped beams

In this section we will show that it is possible to transfer some of the $im\Gamma$ to the muon neutrino propagator while keeping a clear conscience. This is in the context op pancake-shaped beams. In this case, we assume the transversal beam size to be much larger than the decay length $1/\Gamma$, while the longitudinal beam size is much smaller than the decay length. We do this by using momenta that have a large width in the longitudinal direction and a small width in the transversal direction. We assume that the unstable particle is produced at a position away from the position where it collides. We do this by translating the particle having momentum p_2 in (1) an invariant distance s (as seen by the p_1-particle) away from the origin. This is done by a adding factor $e^{isp_1\cdot p_2'/m_1}$ in the matrix element. The complex conjugated matrix element gets a factor $e^{-isp_1\cdot p_2''/m_1}$. The variables p_2' and p_2'' are integration variables in momentum space. They integrate over the momentum distribution.

Furthermore, we take into account (as we have done for the case of infinitely wide beams) the fact that unstable particles cannot really be in/out states. The real matrix element describing the collision process includes the production of the unstable particle. The incoming unstable particle differs from a stable one in that its momentum need not be on its mass shell. The momentum distribution is determined by its propagator $((p_1')^2 - m_1^2 + im_1\Gamma)^{-1}$, which makes sure that the momentum is peaked around the mass shell.

We consider the quantity F which contains the factors of the matrix element that play a role in making the cross section finite. These are the factors that are peaked sharp enough to notice that p_1' (and p_2', of course) do not have a definite value but are peaked around the value p_1. It is given by

$$F = \frac{e^{isp_2'\cdot p_1/m_1}}{(p_1')^2 - m_1^2 + im_1\Gamma}\frac{1}{(q_2 - p_2')^2 - M^2 + i\epsilon}\ . \qquad (27)$$

We now integrate over the value of $(p_1')^2$. The integration path is chosen such that outgoing momenta are kept fixed, while the change Δ^μ of the incoming momenta is taken to be a linear combination of these momenta. Thus momenta are parameterized by

$$\begin{aligned}(p_1')^\mu(t) &= (p_1')^\mu(0) + t\Delta^\mu;\\ (p_2')^\mu(t) &= (p_2')^\mu(0) - t\Delta^\mu;\\ (k')^\mu(t) &= (k')^\mu(0) + t\Delta^\mu,\end{aligned} \qquad (28)$$

where t is just a parameter and has nothing to do with the t-channel. Furthermore we should demand that the value of $(p_2')^2$ is kept fixed. It is not very difficult to think of a scatter setup where the particle with momentum p_2 is a stable one, so we had indeed better not vary this one. These demands are satisfied by taking

$$\Delta^\mu = \frac{1}{2}\frac{(p_1\cdot p_2)p_2^\mu - p_2^2 p_1^\mu}{(p_1\cdot p_2)^2 - p_1^2 p_2^2}\ . \qquad (29)$$

We take a linear combination of p_1 and p_2 because of the pancake shape. These momenta only have a sufficiently large spread in momentum space in the longitudinal direction, so this direction should be chosen for the integration path. Filling this into (27) and using contour integration over t, we find that for the matrix element the integration boils down to the substitution

$$\begin{aligned}F \to &- 2\pi i\frac{e^{isp_2\cdot p_1/m_1 - s\Gamma/2}}{k^2 - M^2 - i\alpha m_1\Gamma}\delta(p_1^2 - m_1^2)\\ &- 2\pi i\frac{e^{isp_2\cdot p_1/m_1}}{p_1^2 - m_1^2 + im_1\Gamma}\theta(\alpha)\delta\left(\alpha^{-1}(k^2 - M^2)\right),\end{aligned} \qquad (30)$$

where $\alpha = 2\Delta\cdot q_2$.

We see that this no longer causes a divergence, so we will square this and work out the Golden Rule to find a cross section. The square of both terms contains a Breit-Wigner of which the square can be approximated by a δ function. However, before we do this, we should discuss the meaning of the two terms. In the first one we effectively first integrate over the width of the unstable particle and then over the width of the decay product. In the second term it is the other way around. The interpretation is that the width that is integrated over first corresponds to the smallest virtuality (this corresponds to the largest distance scale). Thus, the first term describes events where the unstable particle does not decay or decays nearby the spot where the collision happens. The second term describes events where the unstable particle decays long before the decay product collides. We interpret this as the case where the unstable particle is far off shell and is never really produced but already decays during the production process. We decide to drop the second term and keep the first one. If we square the term that we decided to keep, we get the familiar decay law (i.e., the $e^{-\Gamma s}$).

The consequence of the just described procedure is that our squared matrix element contains a factor

$$\frac{1}{(k^2 - M^2)^2 + \alpha^2 m_1^2\Gamma_1^2} \sim \frac{\pi}{\alpha m_1\Gamma}\delta(k^2 - M^2). \qquad (31)$$

For the cross section, we find

$$\begin{aligned}\sigma(p_1, p_2) = \int \tilde{d}k\, &\frac{\sqrt{(p_2\cdot k)^2 - p_2^2k^2}}{\sqrt{(p_1\cdot p_2)^2 - p_1^2p_2^2}}\frac{e^{-s\Gamma}}{\alpha}\\ &\times\sigma(p_2, k)f_s(k|p_1).\end{aligned} \qquad (32)$$

The quotient of flux factors is the generalization of the $1 - \cos\theta$ in (14). The only factor that we might not have expected is the $1/\alpha$.

We can shed light on this factor $1/\alpha$ in the rest frame of the stable particle (the one with momentum p_2). In that frame we have, writing out all inner products,

$$\sigma(p_1, p_2) = \int \tilde{d}k\, \frac{e^{-s\Gamma}}{|\cos\theta|}\sigma(p_2, k)f_s(k|p_1). \qquad (33)$$

We see that the flux factor together with the $1/\alpha$ turn into a factor $1/\cos\theta$. This is because we are describing the collision of two pancake-shaped beams. The p_2 particle is taken to be in its rest frame here, so it should be imagined to be a pancake that does not move. The decay product that collides with a particle in the pancake emerges under an angle θ. Because of this, it sees the thickness of the pancake enlarged by a factor $1/\cos\theta$.

5 Conclusions

The problem that the cross section diverges for colliding unstable particles can be considered for the case of beams much smaller than the decay length or for beams much larger than the decay length. We also had a look at what happens if two pancake-shaped beams collide.

In the first case, the divergent part of the cross section is proportional to the transverse beam size. We can confirm the result of [2] that for $\sqrt{s} \approx 100\,\mathrm{GeV}$ the part of the cross section independent of the beam size and the part proportional to it are of the same order of magnitude. However, for higher energies ($\sqrt{s} > 150\,\mathrm{GeV}$ already) the part proportional to the beam size is at least three orders of magnitude smaller. At still higher energies this becomes even more, so for high energy colliders the linear beam size effect can safely be ignored. In practice this can be done by imposing cuts as one would have in a collider.

In the second case, it should not be "solved" by introducing complex momentum components. This artifically introduces a decay time for the decay product, which does not seem to be a real physical effect. The authors of this paper are not aware of the existence of a law of "conservation of decay width". The result that is obtained by the introduction of complex momentum co-ordinates can (if we allow for a not unreasonable modification in the definition of the cross section) also be obtained by using our methods. Our methods involve considering the wave function of the unstable particle to be a function of four momentum components. This is reasonable because the unstable particle cannot be an in/out state and really is an internal line of a bigger Feynman diagram. After this, the reasoning proceeds along the same lines as for the case of realistic beams. We find (if we do not allow for the just mentioned modification and assume cylindrically shaped beams) the same linear beam size effect.

Another method, namely integrating over the width of p^2 of the unstable particle, is the right thing to do for the case of pancake-shaped wave packets.

One might ask to what extent delicate gauge cancellations are destroyed by our approach. We do not expect these problems to be qualitatively worse than those encountered in, say, calculating loop corrections to LEP-2 processes (cf. [8]). This point will be addressed in a forthcoming publication.

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